CENTRAL ELEMENTS FOR QUANTUM AFFINE ALGEBRAS AND AFFINE MACDONALD'S OPERATORS

Pavel I. Etingof

Department of Mathematics
Harvard University
Cambridge, MA 02138, USA
e-mail: etingof@math.harvard.edu

December 19, 1994

Abstract

We describe a generalization of Drinfeld's description of the center of a quantum group to the case of quantum affine algebras. We use the obtained central elements to construct the affine analogue of Macdonald's difference operators.

1. The center of $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is a simple Lie algebra.

1.1. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ of rank r. Let $\mathfrak{h}$ be a Cartan subalgebra in $\mathfrak{g}$. Let W be the Weyl group. We fix a Weyl group invariant inner product $\langle,\rangle$ on $\mathfrak{h}^*$ and $\mathfrak{h}$ by setting $\langle\alpha,\alpha\rangle = 2$ for short roots. Fix a polarization of $\mathfrak{g}$. Let $\rho \in \mathfrak{h}^*$ be the half-sum of positive roots. Let Q be the root lattice of $\mathfrak{g}$, Q^+ be the semigroup with 0 spanned by the positive roots. Let $\alpha_1, ..., \alpha_r$ be the simple roots. Let $A = (a_{ij})$, $a_{ij} = 2\langle\alpha_i,\alpha_j\rangle/\langle\alpha_i,\alpha_i\rangle$ be the Cartan matrix of $\mathfrak{g}$. Let P be the weight lattice of $\mathfrak{g}$, P^+ be the set of dominant integral weights, and $\omega_1, ..., \omega_r$ be the fundamental weights. Let N be the order of P/Q. Note that for any $\lambda, \mu \in P$ $N\langle\lambda,\mu\rangle \in \mathbb{Z}$.

1.2. Let $\hat{q}$ be a formal variable. Let $q = \hat{q}^N$. For any $a \in \frac{1}{N}\mathbb{Z}$ we define $q^a := \hat{q}^{Na}$.

Let $U_q(\mathfrak{g})$ be the quantum group corresponding to $\mathfrak{g}$ ([Dr1,J1]). It is a Hopf algebra generated over the field $F = \mathbb{C}(\hat{q})$ by the elements E_i, F_i, $1 \le i \le r$, and

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

K_β, $\beta \in P$, freely modulo the relations:

$$K_\beta K_\gamma = K_{\beta+\gamma}, \ K_0 = 1, \ K_\beta E_i K_{-\beta} = q^{\langle \beta, \alpha_i \rangle} E_i,$$

$$K_\beta F_i K_{-\beta} = q^{-\langle \beta, \alpha_i \rangle} F_i, \ [E_i, F_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}, \ q_i = q^{\frac{1}{2}\langle \alpha_i, \alpha_i \rangle}, k_i = K_{\alpha_i},$$

$$\sum_{l=0}^{1-a_{ij}} (-1)^l \begin{bmatrix} 1 - a_{ij} \\ l \end{bmatrix}_{q_i} E_i^{1-a_{ij}-l} E_j E_i^{l-1} = 0, i \neq j,$$

$$\sum_{l=0}^{1-a_{ij}} (-1)^l \begin{bmatrix} 1 - a_{ij} \\ l \end{bmatrix}_{q_i} F_i^{1-a_{ij}-l} F_j F_i^{l-1} = 0, i \neq j,$$

where $\begin{bmatrix} m \\ l \end{bmatrix}_q := \prod_{j=0}^{l-1} \frac{[m-j]_q}{[l-j]_q}$, $[m]_q := \frac{q^m - q^{-m}}{q - q^{-1}}$.

The comultiplication is defined by $\Delta(E_i) = E_i \otimes k_i + 1 \otimes E_i$, $\Delta(F_i) = F_i \otimes 1 + k_i^{-1} \otimes F_i$, $\Delta(K_\beta) = K_\beta \otimes K_\beta$. The antipode is given by $S(E_i) = -E_i k_i^{-1}$, $S(F_i) = -k_i F_i$, $S(K_\beta) = K_{-\beta}$. The counit ε annihilates E_i and F_i and maps K_β to the identity.

1.3. Let K be an extension of F. We call any character $\theta : P \to K^*$ a K-weight for $U_q(\mathfrak{g})$. We say that a representation V of $U_q(\mathfrak{g}) \otimes K$ is with highest weight θ if it is generated by a vector v such that $E_i v = 0$, $1 \leq i \leq r$, and $K_\beta v = \theta(\beta)v$, $\beta \in P$. Note that any character $\theta : P \to K^*$ extends to a ring homomorphism $\mathbb{C}[P] \to K$. We denote this homomorphism also by θ.

In particular, for $\mu \in P$ we can define $\theta_\mu : P \to F^*$ by $\theta_\mu(\lambda) = q^{\langle \mu, \lambda \rangle}$.

1.4. Let $\mathcal{O}_f$ be the category of finite-dimensional representations V of $U_q(\mathfrak{g})$ such that $V = \oplus_{\lambda \in P} V[\theta_\lambda]$, $K_\beta|_{V[\theta_\lambda]} = q^{\langle \beta, \lambda \rangle} Id$. For $\mu \in P^+$, let $V_\mu \in \mathcal{O}_f$ be the irreducible representation with highest weight θ_μ.

1.5. Let $U_q(\mathfrak{n}^\pm)$ be the subalgebras in $U_q(\mathfrak{g})$ generated by $\{E_i\}$, $\{F_i\}$, respectively. Let $\{a_i, i \geq 0\}$ be a homogeneous basis of $U_q(\mathfrak{n}^+)$ ($a_0 = 1$), and let $\{a_i^*\}$ denote the dual basis to $\{a_i\}$ with respect to the Drinfeld pairing between $U_q(\mathfrak{n}^+)$ and $U_q(\mathfrak{n}^-)$ [Dr1].

1.6. Let $V \in \mathcal{O}_f$. Define an element $C_V \in U_q(\mathfrak{g})$ by

(1) $$C_V := \sum_{i,j,\lambda} q^{2\langle \lambda, \rho \rangle} \mathrm{Tr}|_{V[\theta_\lambda]}(a_i a_j^*) a_i^* K_{2(\lambda - \beta_j)} a_j,$$

where β_i is the weight of a_i, i.e. $K_\gamma a_i K_{-\gamma} = q^{\langle \gamma, \beta_i \rangle} a_i$, $\gamma \in P$. Clearly, the sum in (1) is finite: almost all terms vanish.

Remark. The element C_V can be shortly written in the form

$$\mathrm{Tr}|_V(R_{21} R(1 \otimes K_{2\rho})),$$

where R is the universal R-matrix for $U_q(\mathfrak{g})$ [Dr1], and R_{21} is the result of permutation of the two components of R.

1.7.

Theorem 1. [Dr1],[R]

(i) C_V belongs to the center $\mathcal{Z}$ of $U_q(\mathfrak{g})$.

2

(ii) (quantum Harish-Chandra's theorem) Let Rep$\mathfrak{g}$ be the Grothendieck ring of the category $\mathcal{O}_f$ as a tensor category. Then the map $V \to C_V$ is a ring isomorphism $F \otimes_{\mathbb{Z}} Rep\mathfrak{g} \to \mathcal{Z}$. Thus, $\{C_{V_\mu}, \mu \in P^+\}$ is a basis of $\mathcal{Z}$, and $\mathcal{Z} = F[C_{V_{\omega_1}}, ..., C_{V_{\omega_r}}]$.

(iii) (formula for the Harish-Chandra homomorphism) Let K be an extension of F and M be a highest weight representation of $U_q(\mathfrak{g}) \otimes K$ with highest weight $\sigma : P \to K^*$. Then $C_V|_M = \theta(\chi_V)$, where $\theta = \sigma^2 \theta_{2\rho}$, and $\chi_V \in \mathbb{C}[P]$ is the character of V.

2. Generalization to the affine case.

2.1. Let $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$ be the affine Lie algebra corresponding to $\mathfrak{g}$. Let $\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$ be the Cartan subalgebra in $\mathfrak{g}$. Then $\tilde{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\varepsilon \oplus \mathbb{C}\delta$, so that $\varepsilon(c) = \delta(d) = 1$, and $\delta|_{\mathfrak{h} \oplus \mathbb{C}c} = \varepsilon|_{\mathfrak{h} \oplus \mathbb{C}d} = 0$. We can extend the form $\langle,\rangle$ to a nondegenerate form on $\tilde{\mathfrak{h}}^*$ and $\tilde{\mathfrak{h}}$ by setting $\langle \varepsilon, \delta \rangle = 1$, and $\langle \varepsilon, \varepsilon \rangle = \langle \delta, \delta \rangle = \langle \varepsilon, \mathfrak{h}^* \rangle = \langle \delta, \mathfrak{h}^* \rangle = 0$. Define $\alpha_0 = \delta - \tilde{\alpha}$, where $\tilde{\alpha}$ is the maximal root of $\mathfrak{g}$. Then the simple roots of $\tilde{\mathfrak{g}}$ are $\alpha_0, ..., \alpha_r$. The affine Weyl group $\hat{W}$ is generated by the finite Weyl group W and an additional reflection s_0 acting by $s_0\lambda = \lambda - \langle \lambda, \alpha_0 \rangle \alpha_0$. It acts in a natural way on $\tilde{\mathfrak{h}}^*$. This action preserves $\langle,\rangle$.

Let the weight $\hat{\rho} \in \tilde{\mathfrak{h}}^*$ be defined by the condition $\langle \hat{\rho}, \alpha_i \rangle = \frac{1}{2} \langle \alpha_i, \alpha_i \rangle$ for all simple roots α_i, $i = 0, ..., r$, and $\langle \hat{\rho}, \varepsilon \rangle = 0$. Then $\hat{\rho} = \rho + g\varepsilon$, where g is the dual Coxeter number of $\mathfrak{g}$.

Let $\hat{\omega}_j \in \tilde{\mathfrak{h}}^*$, $0 \le j \le r$, be the fundamental weights of $\tilde{\mathfrak{g}}$, defined by $\langle \hat{\omega}_j, \alpha_i \rangle = \frac{\delta_{ij}}{2} \langle \alpha_i, \alpha_i \rangle$, and $\langle \hat{\omega}_j, \varepsilon \rangle = 0$. Define $\tilde{P}$ to be the set of all weights of the form $\lambda = m\delta + \sum_{j=0}^r m_j \hat{\omega}_j$, $m, m_j \in \mathbb{Z}$. Let $\tilde{P}^+$ be the set of dominant integral weights, i.e. weights of the form $\lambda = m\delta + \sum_{j=0}^r m_j \hat{\omega}_j$, $m \in \mathbb{Z}, m_j \in \mathbb{Z}_+$. Let $\tilde{Q}$ be the root lattice of $\tilde{\mathfrak{g}}$, and let $\tilde{Q}^+$ be the semigroup with 0 spanned by the positive roots.

2.2. Let $U_q(\tilde{\mathfrak{g}})$ be the quantum affine algebra corresponding to $\tilde{\mathfrak{g}}$ ([Dr1]). It is defined as follows. Its generators (over F) are E_i, F_i, $i = 0, ..., r$, $K_\beta, \beta \in \tilde{P}$, which satisfy exactly the same relations as in Section 1, with the only difference that i, j vary from 0 to r, and β varies over $\tilde{P}$. Let $U_q(\hat{\mathfrak{g}})$ be the subalgebra in $U_q(\tilde{\mathfrak{g}})$ generated by $E_i, F_i, K_\beta, \beta \in \tilde{P}, \langle \beta, \delta \rangle = 0$.

Let $\{e^\mu, \mu \in \tilde{P}\}$ be the natural basis of $\mathbb{C}[\tilde{P}]$. Let $F[[\tilde{P}]]_T$ denote the ring of formal series $\sum_{m=0}^\infty c_m e^{\lambda_m}$, $c_m \in F$, $\lambda_m \in \tilde{P}$, such that $\langle \lambda_m, \hat{\rho} \rangle \to -\infty$, $m \to \infty$, and the order of the pole of the function c_m at $\hat{q} = 0$ and $\hat{q} = \infty$ is bounded from above as a function of m. We call such formal series *tempered* (this is why we use the subscript T in the notation). Let A be the intersection of all the translates of $F[[\tilde{P}]]_T$ under the action of the affine Weyl group. (It is easy to see that $\hat{W}$ does not preserve $F[[\tilde{P}]]_T$.) For $\phi = \sum c_m e^{\lambda_m} \in A$, we formally set $\xi(\phi) = \sum c_m q^{\langle \lambda_m, \hat{\rho} \rangle} K_{\lambda_m}$.

Let $U_q(\hat{\mathfrak{n}}^\pm)$ be the subalgebras in $U_q(\tilde{\mathfrak{g}})$ generated by $\{E_i\}$, $\{F_i\}$, respectively. Let $\{a_i, i \ge 0\}$ be a homogeneous basis of $U_q(\hat{\mathfrak{n}}^+)$ ($a_0 = 1$), and let $\{a_i^*\}$ denote the dual basis to $\{a_i\}$ with respect to the Drinfeld pairing between $U_q(\hat{\mathfrak{n}}^+)$ and $U_q(\hat{\mathfrak{n}}^-)$.

Let $\widehat{U_q(\tilde{\mathfrak{g}})}$ denote the completion of $U_q(\tilde{\mathfrak{g}})$ consisting of all formal series of the form $\sum_{i,j} a_i^* \xi(\phi_{ij}) a_j$, where $\phi_{ij} \in A$, and the number of distinct weights of terms in the sum is finite (so that $\widehat{U_q(\tilde{\mathfrak{g}})}$ is a $\tilde{P}$-graded algebra). The algebra structure on $\widehat{U_q(\tilde{\mathfrak{g}})}$ is obvious.

2.3. Let K be an extension of F. We call any character $\theta : \tilde{P} \to K^*$ a K-weight

for $U_q(\tilde{\mathfrak{g}})$. We say that a representation V of $U_q(\mathfrak{g}) \otimes K$ is with highest weight θ if it is generated by a vector v such that $E_i v = 0$, $0 \leq i \leq r$, and $K_\beta v = \theta(\beta)v$, $\beta \in \tilde{P}$. In particular, for $\mu \in \tilde{P}$ we can define $\theta_\mu : \tilde{P} \to F^*$ by $\theta_\mu(\lambda) = q^{\langle \mu, \lambda \rangle}$.

Now assume that $K = L(s)$, where s is a transcendental element over a field $L \supset F$, and $\theta(\alpha_j) \in L[s^{-1}]$, $\theta(\delta) \in s^{-1}L[s^{-1}]$. We call such a weight admissible. Then the map θ extends to an algebra homomorphism $A \to L((s))$. Indeed, if $a = \sum c_m e^{\lambda_m} \in A$ then $\langle \lambda_m, \varepsilon \rangle \to -\infty$, $m \to \infty$, so $\theta(a)$ converges in $L((s))$. Note that the set of admissible weights is invariant under shifting by roots of $\tilde{\mathfrak{g}}$.

Moreover, if $\theta = \theta_\mu$, $\mu \in \tilde{P}$, and $\langle \mu, \delta \rangle > 0$, then θ extends to a ring homomorphism $A \to F_-$, $F_- = \mathbb{C}((\hat{q}^{-1}))$. It follows from the fact that for such μ we have $\langle \mu, \lambda_m \rangle \to -\infty$, $m \to \infty$. Similarly, if $\langle \mu, \delta \rangle < 0$ then θ extends to a ring homomorphism $A \to F_+$, $F_+ = \mathbb{C}((\hat{q}))$. For convenience we denote by $\tilde{P}(\pm)$ the subsets of $\tilde{P}$ consisting of all $\mu \in \tilde{P}$ such that $\langle \mu, \delta \rangle \in \mathbb{N}^\pm$.

2.4. Let M be a module over $U_q(\tilde{\mathfrak{g}}) \otimes K$, where K is an extension of F. For any weight $\theta : \tilde{P} \to K^*$ we denote by $M[\theta]$ the space of all vectors of weight θ in M. We say that M has weight decomposition if $M = \oplus_\theta M_\theta$. We say that M is smooth if it has a weight decomposition, and $U_q(\hat{\mathfrak{n}}^+)v$ is a finite-dimensional space for any $v \in M$. Let $\mathcal{S}_K$ be the category of smooth modules. Let $\mathcal{S}_L^a$ be the full subcategory in $\mathcal{S}_{L(s)}$ of smooth modules having only admissible weights, and $\mathcal{S}_\pm$ be the full subcategory of $\mathcal{S}_F$ consisting of modules having only weights of the form θ_μ, $\mu + \hat{\rho} \in \tilde{P}(\pm)$. Then, as follows from Section 2.3, for any $M \in S_L^a$, $M \otimes L((s))$ naturally extends to a module over $\widehat{U_q(\tilde{\mathfrak{g}})} \otimes L((s))$, and for any $M \in \mathcal{S}_\pm$, $M \otimes F_\mp$ naturally extends to a module over $\widehat{U_q(\tilde{\mathfrak{g}})} \otimes F_\mp$. The action of $\xi(A)$ in these extensions are defined by $\xi(\phi)|_{M[\theta]} = \theta(\phi)\theta_{\hat{\rho}}(\phi)\mathrm{Id}$.

2.5. We would like to construct some central elements in $\widehat{U_q(\tilde{\mathfrak{g}})}$. One of these elements is obvious – it is K_δ. The rest of the central elements can be constructed analogously to Drinfeld's method.

Let $U_q(\tilde{\mathfrak{g}})_2$ be the subalgebra in $U_q(\tilde{\mathfrak{g}})$ generated by $U_q(\hat{\mathfrak{g}})$ and K_ε^2. It is clear that $U_q(\tilde{\mathfrak{g}})_2$ is a Hopf subalgebra, and $U_q(\tilde{\mathfrak{g}}) = U_q(\tilde{\mathfrak{g}})_2 \oplus K_\varepsilon U_q(\tilde{\mathfrak{g}})_2$. Let $\tilde{P}_2$ be the sublattice of $\tilde{P}$ spanned by $\hat{\omega}_i$ and 2δ, and let $\tilde{P}_{1/2}$ be the lattice of $\tilde{P}$ spanned by $\hat{\omega}_i$ and $\frac{1}{2}\delta$. If $\lambda \in P_{1/2}$, we define the character $\theta_\lambda : P_2 \to F^*$ by the usual formula $\theta_\lambda(\beta) = q^{\langle \lambda, \beta \rangle}$.

Let $\mathcal{O}_i$ be the category of integrable representations V of $U_q(\tilde{\mathfrak{g}})_2$ such that $V = \oplus_{\lambda \in \tilde{P}_{1/2}} V[\theta_\lambda]$, $\dim V[\theta_\lambda] < \infty$, and $K_\beta|_{V[\theta_\lambda]} = q^{\langle \beta, \lambda \rangle}Id$, $\beta \in \tilde{P}_2$. (Integrability means that the restriction of V to any $U_q(sl_2)$-subalgebra corresponding to a simple root is a direct sum of finite dimensional modules). This category is semisimple ([L]): any object is a (possibly infinite) direct sum of irreducible objects.

For $\mu \in P_{1/2}$, let $V_\mu \in \mathcal{O}_i$ be the irreducible representation of $U_q(\tilde{\mathfrak{g}})_2$ with highest weight θ_μ. Any simple object in $\mathcal{O}_i$ is isomorphic to V_μ for a unique μ. Characters of representations $V \in \mathcal{O}_i$ are infinite linear combinations of e^μ, $\mu \in \tilde{P}_{1/2}$.

Clearly, $\mathcal{O}_i$ is closed with respect to the tensor product.

2.6. Let $V \in \mathcal{O}_i$. Define an element $C_V \in U_q(\tilde{\mathfrak{g}})$ by

$$(2) \qquad C_V := \sum_{i,j,\lambda} q^{2\langle \lambda, \hat{\rho} \rangle} \mathrm{Tr}|_{V[\theta_\lambda]}(a_i a_j^*) a_i^* K_{2(\lambda - \beta_j)} a_j,$$

where β_i is the weight of a_i. Clearly, the sum in (2) is a well defined element of

4

$\widehat{U_q(\tilde{\mathfrak{g}})}$. For example, if V_n is the 1-dimensional $U_q(\tilde{\mathfrak{g}})_2$-module in which $U_q(\hat{\mathfrak{g}})$ acts trivially and K_ε^2 acts by multiplication by q^n then $C_{V_n} = q^{gn} K_\delta^n$.

Remark. 1. The element C_V can be shortly written in the form

$$\mathrm{Tr}|_V (R_{21} R (1 \otimes K_{2\hat\rho})),$$

where R is the universal R-matrix for $U_q(\tilde{\mathfrak{g}})$ [Dr1], and R_{21} is the result of permutation of the two components of R.

2. The reason we need to introduce the algebra $U_q(\tilde{\mathfrak{g}})_2$ is the following. If we only consider elements C_V with V being an integrable $U_q(\tilde{\mathfrak{g}})$-module, like we did in the finite-dimensional case, then the algebra generated by all such element will not contain K_δ (although it will contain its square).

2.7. The following theorem is the affine analogue of Drinfeld's theorem.

Theorem 2. *(i) C_V belongs to the center $\tilde{\mathcal{Z}}$ of $\widehat{U_q(\tilde{\mathfrak{g}})}$.*

(ii) Let $\mathrm{Rep}\tilde{\mathfrak{g}}$ be the Grothendieck ring of the category $\mathcal{O}_i$ as a tensor category. Then the map $V \to C_V$ is a ring isomorphism $F \otimes_{\mathbb{Z}} \mathrm{Rep}\tilde{\mathfrak{g}} \to \tilde{\mathcal{Z}}$. Thus, $\{C_{V_\mu}, \mu \in \tilde{P}^+\}$ is a topological basis of $\tilde{\mathcal{Z}}$, and $\tilde{\mathcal{Z}} = F[C_{V_{\hat\omega_0}}, ..., C_{V_{\hat\omega_r}}]((K_\delta))_T$ (here, as before, the subscript T means the tempered series, i.e. such that the order of poles of of its coefficients at $\hat{q} = 0$ and $\hat{q} = \infty$ are bounded from above).

(iii) Let L be an extension of F, $K = L(s)$ be the field of rational functions of s. Let M be a highest weight representation of $U_q(\tilde{\mathfrak{g}}) \otimes K$ with an admissible highest weight $\sigma : \tilde{P} \to K^*$. Then $M \otimes L((s))$ extends to a representation of $\widehat{U_q(\tilde{\mathfrak{g}})} \otimes L((s))$, and C_V acts there as $\theta(\chi_V) Id$, where χ_V is the character of V, and $\theta = \sigma^2 \theta_{2\hat\rho}$.

(iv) Let $\nu \in \tilde{P}$ and $\langle \nu, \delta \rangle + g \in \mp \mathbb{N}$. Let M be any representation of $U_q(\tilde{\mathfrak{g}})$ with highest weight ν. Then $M \otimes F_\pm$ extends to a representation of $\widehat{U_q(\tilde{\mathfrak{g}})} \otimes F_\pm$ in a natural way, and C_V acts there as $\theta_{2(\nu+\hat\rho)}(\chi_V) Id$, where χ_V is the character of V.

2.8. **Remarks.** 1. The statement about topological basis in (ii) means that any element $z \in \tilde{\mathcal{Z}}$ can be uniquely represented as $\sum_n b_n C_{V_{\mu_n}}$, $b_n \in F$, and $\langle \mu_n, \hat\rho \rangle \to -\infty$ as $n \to \infty$.

2. We formulate (iv) separately from (iii) because weights in $\tilde{P}$ are not admissible.

3. This theorem can be easily generalized to the case of $U_q(\mathfrak{a})$ – quantization of an arbitrary symmetrizable Kac-Moody algebra $\mathfrak{a}$. The changes which need to be made are obvious, and we do not discuss them here.

4. In the classical case $q = 1$, Kac [K] constructed central elements in a completion of $U(\mathfrak{a})$, where $\mathfrak{a}$ an arbitrary symmetrizable Kac-Moody algebra. Kac's construction assigns such a central element to any function on the Tits cone in the set of weights, in such a way that the this assignment inverts the Harish-Chandra homomorphism. Kac's construction easily generalizes to quantum groups, and the elements C_V are examples of central elements one can obtain using this construction. But in the quantum case we also have nice explicit formulas like (1),(2), which are absent in the classical case.

2.9. Proof of Theorem 2. The proof is analogous to the proof of Theorem 1.

(i) Let V_1, V_2 be highest weight representations of $U_q(\tilde{\mathfrak{g}})$, and let $X : V_1 \otimes V_2 \to V_1 \otimes V_2$ be an intertwining operator. Assume that the trace $\mathrm{Tr}|_{V_2}(X(1 \otimes K_{2\hat\rho}))$ makes sense. Then it defines an intertwining operator $V_1 \to V_1$, i.e. commutes with

the action of $U_q(\tilde{\mathfrak{g}})$. Therefore, (i) follows from Remark 2.6 and the fact that $R_{21}R$ commutes with $\Delta(a)$, $a \in U_q(\tilde{\mathfrak{g}})$.

(iii),(iv). When we apply C_V to the highest weight vector $v \in M$, all terms in (2) with $i,j > 0$ vanish, and what remains is

$$C_V v = \sum_\lambda q^{\langle 2\lambda, \hat{\rho} \rangle} \dim V[\theta_\lambda] K_{2\lambda} v = \theta(\chi_V)v.$$

Since M is generated by v, we get (iii),(iv).

(ii) The fact that the assignment $V \to C_V$ is a ring homomorphism follows from the fusion property of the R-matrix. Moreover, Kac [K] showed that a central element $C \in \widehat{U_q(\tilde{\mathfrak{g}})}$ of the form $\sum a_i^* \xi(\phi_{ij})a_j$ is completely determined by ϕ_{00} (strictly speaking, Kac's paper treats the case $q = 1$, but the results we need are obviously valid in the quantum case). Thus, the map $V \to C_V$ is injective. To show that it is surjective, let us recall [K] that $\theta_\lambda(\phi_{00}) = \theta_{s_\alpha(\lambda)}(\phi_{00})$ when $\langle \lambda + \hat{\rho}, \alpha \rangle = \frac{n}{2}\langle \alpha, \alpha \rangle$, where $n \in \mathbb{N}$, α is a positive root of $\tilde{\mathfrak{g}}$, and s_α is the Weyl reflection with respect to α (the Kac-Kazhdan condition); this is because the Verma module with highest weight θ_λ contains the Verma module with highest weight $\theta_{s_\alpha(\lambda+\hat{\rho})-\hat{\rho}}$ when the Kac-Kazhdan condition is satisfied, and C_V acts by $\theta_{\lambda+\hat{\rho}}(\phi_{00})Id$ on the Verma module with highest weight θ_λ. Therefore, $\phi_{00} \in A^{\hat{W}}$, where $A^{\hat{W}}$ is the set of $\hat{W}$-invariant elements in A. But the algebra $A^{\hat{W}}$ is nothing else but the completed algebra of invariant theta functions associated to the affine root system of $\tilde{\mathfrak{g}}$. By a theorem of Looijenga ([Lo]), this algebra is topologically spanned over F by the characters of integrable representations, so we get (ii). $\square$

Remarks. 1. As we have seen, the construction of the center for $U_q(\hat{\mathfrak{g}})$ given above works at any non-critical value of the central charge: $K_\delta \neq q^{-g}$. At the critical value q^{-g}, the structure of the center is different. In fact, in this case the center is much bigger – it contains an algebra of polynomials of infinitely many variables. However, even in this case the above construction (formula (2)) allows to obtain a central element in a completion of $U_q(\hat{\mathfrak{g}})$ if the module V is taken not from the category $\mathcal{O}_i$ but from the category of finite-dimensional $U_q(\hat{\mathfrak{g}})$-modules. This construction was discovered in [RS]; see also [DE].

2. The constructions of central elements for $U_q(\mathfrak{g})$, $U_q(\hat{\mathfrak{g}})$ by formulas (1),(2), as well as the construction of central elements of $U_q(\hat{\mathfrak{g}})$ at the critical level given in [RS],[DE], mentioned in the previous remark, are special cases of the general construction of categorical trace defined by J.Bernstein [B], as follows.

Let A, B be categories, $F : A \to B$ be a functor, and $F^*, {}^*F : B \to A$ be right and left adjoint functors to F. Let $\varepsilon : F^* \to {}^* F$ be a morphism of functors. Let X be an object in A and $a : FX \to FX$ be an endomorphism. Consider the morphism $tr(a) : X \to X$ defined as the composition of morphisms $tr(a) = i_X \circ^* F(a) \circ \varepsilon_{FX} \circ j_X$, where $i_X : F^*FX \to X$, $j_X : X \to^* FFX$ are the adjunction morphisms. This defines the linear operator of categorical trace $tr : \mathrm{End}(F) \to \mathrm{End}(Id_A)$, where $Id_A : A \to A$ is the identity functor.

Let U be an associative algebra over a ring $\mathcal{F}$, and A be a full subcategory of the category of U-modules. Then the center $Z(U)$ of U naturally maps to the ring $\mathrm{End}(F)$, where F is any functor on A. In particular, if F is the identity functor, this map is often an embedding. Therefore $Z(U)$ can often be identified with a subring of the ring of endomorphisms of the identity functor on the category of representations of U.

6

It is shown in [B] how to use the trace construction to produce many central elements of U when we have only one fixed central element $C \in Z(U)$. Indeed, let $F : A \to A$ be any functor satisfying the above conditions. Then we can consider $tr F(C) \in \text{End}(Id_A)$, and if we are lucky, so that this element is in the image of the center, then we get a new central element $C_F \in Z(U)$. Let us demonstrate how this works for quantum groups.

If U is a qausitriangular Hopf algebra (e.g. $U_q(\mathfrak{g}), U_q(\hat{\mathfrak{g}})$) then one can define an element $u = m((S \otimes 1)(R_{21}))$ in a completion of this algebra, where m is the multiplication map, S is the antipode, and R_{21} is the universal R-matrix with permuted components. Conjugation by u is S^2. This element was defined by Drinfeld [Dr2]. It is easy to show that the element $Z = uS(u)$ is central. Moreover, in the cases of $U_q(\mathfrak{g})$, $U_q(\hat{\mathfrak{g}})$ it is possible to find a central element C such that $\Delta(C) = (C \otimes C)(R_{21}R)$, and $C^2 = Z^{-1}$ (it is obtained by formal extraction of square root from Z^{-1}; Hopf algebras where C exists are called ribbon Hopf algebras, [RT]). Now, if V is an irreducible, integrable highest weight module, and $F = F_V$ is the functor of tensoring of U-modules with V (on the right). Then ${}^*F, F^*$ are the functors of tensoring with ${}^*V, V^*$, respectively, and $u : V^* \to^* V$ defines an isomorphism $\varepsilon : F^* \to^* F$. It is easy to see that $C_V = \frac{1}{C|_V}C^{-1}C_F$, where C_F is the categorical trace of C, and $C|_V$ is the eigenvalue of C in V. This explains formulas (1),(2).

3. Central elements and Macdonald operators

3.1. Macdonald operators acting on the space of rational functions of n variables over $\mathbb{C}(q, t)$ are defined as follows:

$$(3) \qquad M^r = t^{r(r-n)} \sum_{i_1 < i_2 < ... < i_r} \left(\prod_{\substack{j \notin \{i_1...i_r\} \\ l=1...r}} \frac{t^2 x_{i_l} - x_j}{x_{i_l} - x_j} \right) T_{i_1} \ldots T_{i_r}$$

where $(T_i f)(x_1, \ldots, x_n) = f(x_1, \ldots, q^2 x_i, \ldots, x_n)$, t is a formal variable, and $r = 1, \ldots, n-1$.

Macdonald proved [M] that these operators are pairwise commutative: $[M^r, M^s] = 0, 1 \le r, s \le n-1$.

In [EK1],[EK2] it was shown how to obtain Macdonald operators from central elements of the quantum group $U_q(\mathfrak{g})$, where $\mathfrak{g} = \mathfrak{sl}_n$. This was done as follows.

3.2. Recall that fundamental representations of $U_q(\mathfrak{g})$ are q-deformed exterior powers $(\Lambda^r F^n)^*$, $r = 1, ..., n-1$. Consider the elements

$$(4) \qquad C_r := C_{(\Lambda^r F^n)^*} \in U_q(\mathfrak{g}),$$

where $\rho = (\frac{n-1}{2}, \frac{n-3}{2}, ..., \frac{-n+1}{2})$. These elements are central and freely generate the center of $U_q(\mathfrak{g})$, according to Theorem 1.

3.3. Let $\hat{t}$ be a formal variable such that $\hat{t}^2 = t$. Let K be an extension of $F(\hat{t})$, and let $\theta : P \to K^*$ be a weight. Let M_θ denote the Verma module over $U_q(\mathfrak{g})$ with highest weight θ, and let

$$(5) \qquad U = \{p \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}], \deg p = 0\}$$

7

with the action of $U_q(\mathfrak{g}) \otimes K$ given by

(6)
$$E_i \mapsto x_i D_{i+1}, \ F_i \mapsto x_{i+1} D_i,$$

$$K_\beta f(x_1, ..., x_n) = f(q^{\beta_1} x_1, ..., q^{\beta_n} x_n), \beta = (\beta_1, ..., \beta_n) \in P$$

$$(D_i f)(x_1, \dots, x_n) = \frac{tq^{-1} f(x_1, \dots, qx_i, \dots, x_n) - t^{-1} q f(x_1, \dots, q^{-1} x_i, \dots, x_n)}{(q - q^{-1}) x_i}$$

The set of weights of U is the root lattice Q, and every weight subspace is one-dimensional: $U[\mu] = K x_1^{\mu_1} ... x_n^{\mu_n}$, $\mu = (\mu_1, ..., \mu_n) \in Q$. We fix a vector $u \in U[0]$.

3.4.

Lemma 1. *The following two conditions on the Verma module M_θ are equivalent:*
(i) M_θ is irreducible.
(ii) There exists a unique intertwiner

(7)
$$\Phi_\theta \colon M_\theta \to M_\theta \hat{\otimes} U,$$

where $\hat{\otimes}$ denotes the completed tensor product with respect to the principal grading in M_θ, normalized by the condition $\Phi_\theta v = v \otimes u +$ lower order terms, where v is the highest weight vector of M_θ.

Proof of the (i)$\to$ (ii) part is based on the general fact: if a Verma module M_θ is irreducible and U is any $U_q(\mathfrak{g}) \otimes K$-module then the space of intertwiners $M_\theta \to M_\theta \otimes U$ is in one-to-one correspondence with the zero-weight subspace $U[0]$, i.e the space of K_β-invariant vectors in U (for all $\beta \in P$). The (ii)$\to$ (i) part (which we don't use in this paper) follows from the study of singularities of Φ_θ as a function of θ done in [ES]. (In [ES], the case $q = 1$ is treated, but in the quantum case the situation is analogous).

Remark. The module M_θ over $U_q(\mathfrak{g})$ is reducible if and only if the quantum Kac-Kazhdan condition is satisfied: $\theta(2\alpha) = q^{n\langle \alpha, \alpha \rangle}$ for some positive root α of $\mathfrak{g}$, and some integer $n > 0$. This follows from the formula for the determinant of the Shapovalov form, see [DCK].

3.5. Let $Y = \text{Hom}(P, K^*)$. Define a completion $K[[Y]]$ of the group algebra $K[Y]$ of Y. By definition, $K[[Y]]$ is spanned by the subspaces $K[[Y]]_\sigma, \sigma \in Y$, where $K[[Y]]_\sigma$ consists of all, possibly infinite, formal series of the form $\sum_{\beta \in Q^+} c_\beta \sigma \theta_{-\beta}$, where $c_\beta \in K$.

Let M be a highest weight $U_q(\mathfrak{g}) \otimes K$-module. For any $\Phi : M \to M \hat{\otimes} U$ denote by $\Phi[\theta]$ the diagonal block of Φ of weight θ: $\Phi[\theta] : M[\theta] \to M[\theta] \otimes U[0]$. Let $K(Y)$ be the quotient field of $K[Y]$, and let $X : M \otimes K(Y) \to M \otimes K(Y)$ be the linear operator defined by $Xv = \theta v$, $v \in M[\theta]$.

Let $\tau_0 : P \to K^*$ be defined by $\tau_0(\lambda) = \hat{t}^{2\langle \lambda, \rho \rangle}$, and let $\tau = \tau_0 \theta_{-\rho}$.

Set

(8)
$$\Psi_\theta = \text{Tr}|_{M_{\theta\tau} \otimes K(Y)}(\Phi_{\theta\tau} X) := \sum_{\beta \in Q^+} \text{Tr}(\Phi_{\theta\tau}[\theta\tau\theta_{-\beta}]) \theta\tau\theta_{-\beta},$$

This is an element of $K[[Y]] \otimes U[0]$. The construction of Ψ_θ is valid for a generic $\theta \in Y$, i.e. such that the corresponding Verma module $M_{\theta\tau}$ is irreducible.

Since the space $U[0]$ is one-dimensional, we can identify it with K by sending u to 1 and regard Ψ_θ as a series with scalar coefficients, i.e. as an element of $K[[Y]]$.

3.6. Now we explain a connection between Macdonald difference operators and the series Ψ_θ.

Definition. *A standard difference operator is any Laurent polynomial in the operators T_i whose coefficients are rational functions of $y_1 = x_2/x_1, ..., y_{n-1} = x_n/x_{n-1}$ (over $F(\hat{t})$) which are regular at $y_1 = ... = y_{n-1} = 0$.*

It is clear that the algebra of standard difference operators can be embedded in the algebra $K[[y_1, ..., y_{n-1}]][T_1, ..., T_n]$ (by taking the Taylor expansion of the coefficients). It therefore follows that any standard difference operator naturally acts on the space $K[[Y]]_\sigma$, $\sigma \in Y$. This action is given by $T_j \theta = \theta(\omega_j - \omega_{j-1})^2 \theta$ (by convention $\omega_0 = 0$), and $y_j \theta = \theta \theta_{-\alpha_j}$. There is a natural power series topology on $K[[Y]]_\sigma$, and the action of any standard difference operator is continuous in this topology.

3.7.

Theorem 3. *([EK2])*
   (i) For every r between 1 and $n - 1$ there exists a unique standard difference operator $\tilde{M}^r$ defined over $F(t)$, such that

$$(9) \qquad \text{Tr}|_{M_{\theta\tau} \otimes K(Y)}(\Phi_{\theta\tau} C_r X) = \tilde{M}^r \text{Tr}|_{M_{\theta\tau} \otimes K(Y)}(\Phi_{\theta\tau} X)$$

for all θ not satisfying the Kac-Kazhdan condition.
   (ii) $[\tilde{M}^r, \tilde{M}^s] = 0$, $1 \le r, s \le n - 1$.
   (iii) Introduce the series

$$(10) \qquad \varphi = \tau \prod_{i=1}^{\infty} \prod_{\alpha \in R^+} \frac{1 - q^{2i}\theta_{-\alpha}}{1 - q^{2(i-1)}t^2\theta_{-\alpha}} \in AutK[[Y]],$$

where R^+ is the set of positive roots of $\mathfrak{g}$, and τ denotes the operator of multiplication by τ in $K[[Y]]$. Then $\frac{1}{\varphi}\tilde{M}^r(\varphi f) = M^r f$ for any $f \in K[[Y]]$, where M^r are the Macdonald operators defined by (3). Thus, for any $\theta \in Y$ for which $\Psi_{\theta\tau}$ is defined the series $\psi_\theta := \Psi_\theta/\varphi$, is a common eigenvector of the Macdonald operators in $K[[Y]]$: $M^r \psi_\theta = P_r(\theta)\psi_\theta$, where $P_r(\theta) = (\theta\tau_0)^2(\chi_r)$, and χ_r is the character of $(\Lambda^r F^n)^*$.
   (iv) Let $\sigma \in Y$ be such that the operator $\Phi_{\theta\tau}$ exists for any $\theta = \sigma\theta_{-\beta}$, $\beta \in Q^+$. Then the vectors $\psi_{\sigma\theta_{-\beta}}$, $\beta \in Q^+$, form a topological basis of $K[[Y]]_\sigma$ (with respect to the usual power series topology). Thus, M^r are simultaneously diagonalizable in $K[[Y]]_\sigma$. Generically, each of them has a simple spectrum in $K[[Y]]_\sigma$.
   (v) Let $\sigma \in Y$ be a weight such that the weights $\sigma^w := w(\sigma\tau_0)\tau_0^{-1}$, $w \in S_n$, are all distinct, and $\Phi_{\sigma^w \tau}$ exists for each w. Then the system of difference equations $M^r \psi = P_r(\sigma)\psi$, $r \in 1, ..., n - 1$, has $n!$ linearly independent solutions in $\mathcal{K}[[Y]]$. They are ψ_{σ^w}, $w \in S_n$.

3.8. In particular, the construction of functions ψ_θ can be specialized to the case when $\theta = \theta_\lambda$, λ is a weight in P^+. Lemma 1 implies that the operator $\Phi_{\theta\tau}$ always exists in this case, and we have

Theorem 4. *If $\lambda \in P^+$ then the function ψ_{θ_λ} belongs to $\mathbb{C}(q, t)[P]$ and coincides with the Macdonald polynomial $P_\lambda(x; q, t)$ (see [EK2]).*

4. Affine Macdonald operators.

In this section we will define an affine analogue of Macdonald operators using the analogy with Section 3. We consider the case $\mathfrak{g} = sl(n)$.

4.1. Let $p : U_q(\hat{\mathfrak{g}}) \to U_q(\mathfrak{g})$ be the Jimbo homomorphism [J2]. Let $U(1) = p^*(U)$.

Let $\theta : \tilde{P} \to K^*$ be a weight, and let M_θ be the Verma module over $U_q(\tilde{\mathfrak{g}}) \otimes K$ with highest weight θ.

4.2.

Lemma 2. *The following two conditions on the Verma module M_θ are equivalent.*

(i) M_θ is irreducible.

(ii) There exists a unique $U_q(\hat{\mathfrak{g}})$-intertwiner

$$\Phi_\theta \colon M_\theta \to M_\theta \hat{\otimes} U(1),$$

where $\hat{\otimes}$ denotes the completed tensor product with respect to the principal grading in M_θ, normalized by the condition $\Phi_\theta v = v \otimes u +$ lower order terms, where v is the highest weight vector of M_θ.

Proof is analogous to the finite-dimensional case.

Remark. The module M_θ over $U_q(\hat{\mathfrak{g}})$ is reducible if and only if the Kac-Kazhdan condition is satisfied: $\theta(2\alpha) = q^{n\langle \alpha, \alpha \rangle}$ for some root $\alpha > 0$ of $\tilde{\mathfrak{g}}$ and some integer $n > 0$.

4.3. Let $\tilde{Y} = \text{Hom}(\tilde{P}, K^*)$. Define a completion $K[[\tilde{Y}]]$ of the group algebra $K[\tilde{Y}]$ of $\tilde{Y}$. By definition, $K[[\tilde{Y}]]$ is spanned by the subspaces $K[[\tilde{Y}]]_\sigma, \sigma \in \tilde{Y}$, where $K[[\tilde{Y}]]_\sigma$ consists of all, possibly infinite, formal series of the form $\sum_{\beta \in Q^+} c_\beta \sigma \theta_{-\beta}$, where $c_\beta \in K$.

Let M be a highest weight $U_q(\hat{\mathfrak{g}}) \otimes K$-module. For any $\Phi : M \to M \hat{\otimes} U$ denote by $\Phi[\theta]$ the diagonal block of Φ of weight θ: $\Phi[\theta] : M[\theta] \to M[\theta] \otimes U[0]$. Let $K(\tilde{Y})$ be the quotient field of $K[\tilde{Y}]$, and let $X : M \otimes K(\tilde{Y}) \to M \otimes K(\tilde{Y})$ be the linear operator defined by $Xv = \theta v, v \in M[\theta]$.

Let $\tau_0 : \tilde{P} \to K^*$ be defined by $\tau_0(\lambda) = \hat{t}^{2\langle \lambda, \hat{\rho} \rangle}$, and let $\tau = \tau_0 \theta_{-\hat{\rho}}$.

Set

(11) $$\Psi_\theta = \text{Tr}|_{M_{\theta\tau} \otimes K(\tilde{Y})}(\Phi_{\theta\tau} X) := \sum_{\beta \in \tilde{Q}^+} \text{Tr}(\Phi_{\theta\tau}[\theta\tau\theta_{-\beta}])\theta\tau\theta_{-\beta},$$

This is an element of $K[[\tilde{Y}]] \otimes U[0]$. The construction of Ψ_θ is valid for a generic $\theta \in \tilde{Y}$, i.e. such that the corresponding Verma module $M_{\theta\tau}$ is irreducible.

Since the space $U[0]$ is one-dimensional, we can identify it with K by sending u to 1 and regard Ψ_θ as a series with scalar coefficients, i.e. as an element of $K[[\tilde{Y}]]$.

4.4.

Definition. *An affine difference operator is a formal expression of the form $M = \sum a_m T_{\nu_m}, \nu_m \in \tilde{P}$, such that for any $w \in \hat{W}$ $\langle w\nu_m, \hat{\rho} \rangle \to -\infty, m \to \infty$, and $a_m \in F(t)[[\tilde{Y}]]_1$, where 1 denotes the identity character $\tilde{P} \to \mathbb{C}^*$.*

Affine difference operators form an algebra, multiplication in which is defined by $T_\nu \theta = \theta(\nu)^2 \theta T_\nu$. This formula also gives rize to a natural action of the algebra of affine difference operators in $K[[\tilde{Y}]]_\sigma \otimes L((s))$ for any admissible $\sigma \in \tilde{Y}$. Also, if the series M is tempered, in the sense of Section 2, the action of M is defined in $G_\pm[[\tilde{Y}]]_\sigma, \sigma \in \tilde{P}(\mp), G_\pm = \mathbb{C}(\hat{t})((\hat{q}))$. These actions are continuous in the standard power series topology. We will further assume that all highest weights we consider are either admissible or from $\tilde{P}(\pm)$.

4.5.

Theorem 5.

(i) For every r between 0 and $n-1$ there exists a unique affine difference operator $\tilde{M}^{\hat{\omega}_r}$ such that

$$(12) \qquad \mathrm{Tr}(\Phi_{\theta\tau} C_{V_{\hat{\omega}_r}} X) = \tilde{M}^{\hat{\omega}_r} \mathrm{Tr}(\Phi_{\theta\tau} X)$$

for all θ not satisfying the Kac-Kazhdan condition.

(ii) $[\tilde{M}^{\hat{\omega}_r}, \tilde{M}^{\hat{\omega}_s}] = 0$, $0 \le r, s \le n-1$.

The proof of this theorem is analogous to the proof of Theorem 3. Uniqueness follows from the fact that an affine difference operator is uniquely determined by its action in $K[[\tilde{Y}]]_\sigma$ for generic σ, and this action is defined uniquely by (12). Indeed, suppose we have an expression $\sum a_m T_{\nu_m}$ which acts by 0 on $K[[\tilde{Y}]]_\sigma$ for a generic admissible weight σ. In particular, applying it to σ, we get $\sum a_m \sigma(\nu_m)^2 = 0$. This identity for generic admissible σ implies immediately that $a_m = 0$ for all m.

4.6. Introduce the series

$$(13) \qquad \varphi = \tau \prod_{i=1}^{\infty} \prod_{\alpha \in \tilde{R}^+} \frac{1 - q^{2i}\theta_{-\alpha}}{1 - q^{2(i-1)}t^2\theta_{-\alpha}} \in F(t)[[\tilde{Y}]]_1,$$

where $\tilde{R}^+$ is the set of positive roots of $\tilde{\mathfrak{g}}$.

Introduce the affine difference operators $M^{\hat{\omega}_r}$ defined by $M^{\hat{\omega}_r} f = \frac{1}{\varphi} \tilde{M}^{\hat{\omega}_r}(\varphi f)$ for any $f \in K[[\tilde{Y}]]$. These operators are pairwise commutative.

Definition. *We call the operators $M^{\hat{\omega}_r}$ affine Macdonald operators.*

4.7. For any θ for which $\Psi_{\theta\tau}$ is defined consider the series $\psi_\theta := \Psi_\theta / \varphi$.

Corollary. *(i) ψ_θ is a common eigenvector of the Macdonald operators in $K[[\tilde{Y}]]_\theta$: $M^{\hat{\omega}_r}\psi_\theta = \tilde{P}_r(\theta)\psi_\theta$, where $\tilde{P}_r(\theta) = (\theta\tau_0)^2(\chi_{\hat{\omega}_r})$, and $\chi_{\hat{\omega}_r}$ is the character of the $r-$th fundamental representation.*

(ii) Let $\sigma \in \tilde{Y}$ be such that the operator $\Phi_{\theta\tau}$ exists for any $\theta = \sigma\theta_{-\beta}$, $\beta \in \tilde{Q}^+$. Then the vectors $\psi_{\sigma\theta_{-\beta}}$, $\beta \in \tilde{Q}^+$, form a topological basis of $K[[\tilde{Y}]]_\sigma$ (with respect to the usual power series topology). Thus, $M^{\hat{\omega}_r}$ are simultaneously diagonalizable in $K[[\tilde{Y}]]_\sigma$. Generically, each of them has a simple spectrum in $K[[\tilde{Y}]]_\sigma$.

(iii) Let $\sigma \in \tilde{Y}$ be a weight such that the weights $\sigma^w := w(\sigma\tau_0)\tau_0^{-1}$, $w \in \hat{W}$, are all distinct, and $\Phi_{\sigma^w\tau}$ exists for each w. Then the space of solutions of the system of difference equations $M^{\hat{\omega}_r}\psi = \tilde{P}_r(\sigma)\psi$, is topologically spanned by the linearly independent solutions ψ_{σ^w}, $w \in \hat{W}$.

The proof is analogous to the finite-dimensional case.

4.8. In particular, the construction of functions ψ_θ can be specialized to the case when $\theta = \theta_\lambda$, λ is a weight in P^+. Then the weight $\theta\tau$ is admissible ($K = F(s)$, $s = \hat{t}^{-1}$), and Lemma 2 implies that the operator $\Phi_{\theta\tau}$ exists. In this case, we have

Theorem 6. [EK3] *If $\lambda \in P^+$ then the function ψ_{θ_λ} belongs to $\mathbb{C}(\hat{q}, \hat{t}) \otimes_F A^{\hat{W}}$ and coincides with the affine Macdonald polynomial $\hat{J}_\lambda$ defined in [EK3].*

(In fact, it is obvious that the coefficients of $\hat{J}_\lambda$ are in $\mathbb{C}(q, t)$). As shown in [EK3], affine Macdonald polynomials are a basis of $\mathbb{C}(\hat{q}, \hat{t}) \otimes_F A^{\hat{W}}$ triangular with respect to the basis of characters χ_λ. Therefore, we find:

11

Corollary. *The operators $M^{\hat{\omega}_r}$ are $\hat{W}$-invariant.*

This follows from the fact that an affine difference operator is uniquely defined by its action on symmetric theta-functions.

Thus , analogously to the finite-dimensional case, affine Macdonald operators act on the space of symmetric theta-functions and are triangular with respect to the standard ordering of dominant weights. Affine Macdonald polynomials can be defined as their common eigenbasis.

In particular, it follows from this corollary that the coefficients to all T_ν in the affine Macdonald operators are of the form $\sum c_m \theta_{-\alpha_m}$, where $\langle w\alpha_m, \hat{\rho}\rangle \to +\infty$, $m \to \infty$, for all $w \in \hat{W}$. However, these coefficients need not be Weyl group invariant.

References

[B] Bernstein, J., *Trace in categories*, Operator algebras, unitary representations, enveloping algebras, and invariant theory, 1990, pp. 417-424.

[Dr1] Drinfeld, V.G., *Quantum groups*, Proc. Int. Congr. Math., Berkeley, 1986, pp. 798–820.

[Dr2] Drinfeld, V.G., *On almost cocommutative Hopf algebras*, Len. Math. J. **1** (1990), no. 2, 321-342.

[DCK] De Concini, C., and Kac, V., *Quantum groups at roots of 1*, Operator algebras, unitary representations, enveloping algebras, and invariant theory, 1990, pp. 471-508.

[DE] Ding, J., and Etingof, P., *Center of a quantum affine algebra at the critical level*, Math. Res. Let. **1** (1994), no. 4, 469-480.

[EK1] Etingof, P.I. and Kirillov, A.A., Jr, *A unified representation-theoretic approach to special functions, hep-th 9312101*, Functional Anal. and its Applic. **28** (1994), no. 1.

[EK2] Etingof, P.I. and Kirillov, A.A., Jr, *Macdonald's polynomials and representations of quantum groups*, Math. Res. Let. **1** (1994), no. 3, 279-296.

[EK3] Etingof, P.I. and Kirillov, A.A., Jr, *On the affine analogue of Jack and Macdonald polynomials*, to appear in Duke Math. J., hep-th 9403168 (1994).

[ES] Etingof, P., and Styrkas, K., *Algebraic integrability of Schrödinger operators and representations of Lie algebras*, to appear in Compos. Math., hep-th 9403135 (1994).

[J1] Jimbo, M.A., *A q-difference analogue of $U\mathfrak{g}$ and the Yang-Baxter equation*, Lett. Math. Phys. **10** (1985), 62–69.

[J2] Jimbo, M.A., *A q-analogue of $U(\mathfrak{gl}(N+1))$, Hecke algebra and the Yang-Baxter equation*, Lett. Math. Phys. **11** (1986), 247–252.

[K] Kac, V., *Laplace operators of infinite-dimensional Lie algebras and theta functions*, Proc. Nat. Acad. Sci. USA **81** (1984), 645-647.

[L] Lusztig, G., *Introduction to quantum groups*, Birkhauser, Boston, 1994.

[Lo] Looijenga, E., *Root systems and elliptic curves*, Inv. Math. **33** (1976), 17–32.

[M] Macdonald, I.G., *A new class of symmetric functions*, Publ. I.R.M.A. Strasbourg, 372/S-20, Actes 20 Séminaire Lotharingien (1988), 131-171.

[R] Reshetikhin, N.Yu., *Quasitriangle Hopf algebras and invariants of tangles*, Len. Math. J. **1** (1990), 491-513.

[RS] Reshetikhin, N.Yu, and Semenov-Tian-Shansky, M.A., *Central extensions of quantum current groups*, Lett. Math. Phys. **.** **19** (1990), 133-142.

[RT] Reshetikhin, N.Yu., and Turaev, V.G., *Ribbon graphs and their invariants derived from quantum groups*, Invent. Math. **103** (1991), 547-597.